Journal of Applied Mechanics and Technical Physics, Vol. 49, No. 4, pp. 568–579, 2008

STEADY-STATE CONDITIONS OF A NONISOTHERMAL FILM WITH A HEAT-INSULATED FREE BOUNDARY

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UDC 532.516

The equilibrium shapes of a nonisothermal liquid film with a heat-insulated free surface for large Marangoni numbers are investigated in the long-wave approximation using a combination of analytical and numerical methods. It is proved that the two-dimensional problem of the equilibrium of a strip-shaped film has a steady-state solution for an arbitrary large temperature gradient on the boundaries of the strip. An increase in this gradient leads to an abrupt thinning of the film near the heated boundary, which can result in instability and rupture of the film. In the equilibrium problem for a film fixed on a circular contour, the nonuniform distribution of the heat flux on the contour was found to have a significant influence on the free-surface shape.

Key words: *wave flow of liquid, thin nonisothermal liquid film, summation of power series, collocation method.*

Introduction. At present, the theory of wave flow of liquid films has become an independent branch of hydrodynamics with various technological applications [1]. In the analysis of the problems arising in this theory, the main approach is the long-wave approximation. Most efforts in this area have focused on studying isothermal motion of films flowing down a solid surface. Among the papers dealing with nonisothermal flows along an inclined plane, mention should be made of [2–5]. Much less attention has been given to motion of films whose both boundaries are free but nonuniformly heated or contain surfactants [6–10]. The latter case is more difficult to study because among the required functions are both the volumetric and surface concentrations of surfactants. In contrast to concentration, liquid temperature is not divided into volumetric and surface parts, which simplifies the analysis.

A simple example of a free liquid film is water in a sieve mesh. The nonisothermal films formed during coating application and in the manufacture of polymers can be used to design new heat exchange apparatus. We are not aware of experimental studies of free films of the macroscopic scale. It is reasonable to perform such studies on a space station, where the range of steady-state parameters of a free film is wider than under Earth conditions.

The long-wave equations describing the motion of a free weightless film fixed at a plane contour and acted upon by thermocapillary forces were derived in [7, 11]. It turned out that, within the framework of this approximation, the shape of the free boundary of the film can be determined without having detailed information on the dependence of the velocity vector on the vertical coordinate. This distinguishes the problem discussed from the problem of flow of a thin layer of a viscous liquid adjacent to a solid plane. If the film thickness as a function of the coordinates and time is known, the velocity field in the film is found approximately by solving the initial-boundary-value problem formulated in [11].

In the mathematical modeling of nonisothermal film flows, the condition of thermal contact of the liquid and gas phases plays a key role. As a rule, this condition is approximated by the boundary-value condition of the third kind for liquid temperature containing an empirical coefficient. To avoid the use of empirical information, it is necessary to investigate the problem of the joint motion of the liquid and gas (see, for example, [12]) or to examine

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 0021-8944/08/4904-0568

the limiting situations where transfer processes in the gas are not considered. One of such situations corresponds to the assumption of a perfectly heat-conducting free surface [7]. In this case, its temperature coincides with the *a priori* specified gas temperature at the interface. In another case, which is considered in the present paper, the free boundary is heat insulated.

1. Formulation of the Problem. The equilibrium problem for a free weightless film with a heat-insulated free boundary is investigated. In the thin-layer approximation, the behavior of the film is described by two functions: its thickness h and the temperature T averaged over the thickness. The film occupies a plane domain ω, on whose boundary $\partial\omega$ the heat flux is specified. The mathematical formulation of the problem [11] consists of seeking a solution $h(x_1, x_2)$, $T(x_1, x_2)$ of the system of equations

$$\nabla \cdot (h\nabla\Delta h) = \gamma\Delta T, \qquad \nabla \cdot (h\nabla T) = 0, \qquad (x_1, x_2) \in \omega, \tag{1.1}$$

that satisfies the conditions

$$\frac{\partial h}{\partial n} = 0, \qquad h\,\frac{\partial \Delta h}{\partial n} = \gamma\,\frac{\partial T}{\partial n}, \qquad h\,\frac{\partial T}{\partial n} = g(x), \qquad (x_1, x_2) \in \partial\omega; \tag{1.2}$$

$$\int\limits_{\omega} h(x_1, x_2)\, d\omega = S. \tag{1.3}$$

Relations (1.1) and (1.2) are written in dimensionless variables. The average film thickness δ is chosen as the transverse scale of the length, and the diameter l of the physical domain occupied by the film as the longitudinal scale. The temperature scale is the quantity Qlk^{-1} (Q is the characteristic value of the heat flux on the plane contour bounding the film and k is the thermal conductivity of the liquid); $\partial/\partial n$ is the derivative in the direction of the outward normal to the curve $\partial\omega$. Condition (1.3), in which S is the area of the domain ω, implies that the average dimensionless film thickness is equal to unity. The first condition in (1.2) implies that the angle of wetting of the solid cylindrical surface bounding the liquid Σ with cross section $\partial\omega$ is $\pi/2$ (this assumption is not necessary but it simplifies the problem). The function $g(x)$ specifying the distribution of the dimensionless heat flux on the contour $\partial\omega$ satisfies the compatibility condition

$$\int\limits_{\partial\omega} g\, ds = 0, \tag{1.4}$$

where ds is an element of the arc length $\partial\omega$. The second boundary condition in (1.2), derived in [7], follows from the impermeability condition for the surface Σ. The parameter γ, included in this condition is defined by the formula

$$\gamma = \frac{\sigma_T Q l^3}{\sigma_0 k \delta^2},$$

where σ_0 is the average value of the surface tension coefficient $\sigma(T)$ and σ_T is the temperature surface-tension coefficient, and

$$\sigma = \sigma_0 - \sigma_T(T - T_0)$$

(T_0 is the average temperature in the domain ω; the quantities σ_T, σ_0, and T_0 are assumed to be constant and positive). The parameter γ plays the role of the Marangoni number — the main similarity criterion in the theory of thermocapillary flows. We note that, although the paper deals with steady-state states of the film, the velocity field in it is nonzero. However, the problem of calculating the velocity field is not considered here.

It is easy to see that, in the solution of problem (1.1)–(1.3), the function T is determined to within an additive constant. This arbitrariness can be eliminated by requiring that the following condition be satisfied:

$$\int\limits_{\omega} T(x_1, x_2)\, d\omega = 0. \tag{1.5}$$

Next, the normalization condition (1.5) is assumed to be satisfied. In [11], the following fact is proved. If the curve $\partial\omega$ belongs to the Holder class $C^{4+\alpha}$ ($0 < \alpha < 1$) and if the function g belongs to the Holder class $C^{4+\alpha}(\partial\omega)$ and satisfies the compatibility condition (1.4), there exists $\gamma_0 > 0$ such that, for $\gamma \in [0, \gamma_0]$, problem (1.1)–(1.3) has a unique solution $h \in C^{4+\alpha}(\bar{\omega})$, $T \in C^{2+\alpha}(\bar{\omega})$. This solution can be represented as

$$h = 1 + \sum_{k=1}^{\infty} \gamma^k h_k(x_1, x_2), \qquad T = \sum_{k=1}^{\infty} \gamma^k T_k(x_1, x_2). \tag{1.6}$$

Series (1.6) converge in the norms of the spaces $C^{4+\alpha}(\bar{\omega})$ and $C^{2+\alpha}(\bar{\omega})$, respectively, if $\gamma \in [0, \gamma_0]$.

For the case where ω is a unit circle and $g = 2x_1x_2$, the functions h_1, h_2, and T_1 were found in [11]. In the present work, the next terms of the power series (1.6) for this case were calculated. Numerical investigation of the terms obtained allows a prediction of the radius of convergence of the power series. From the results of the numerical calculations, it follows that the film strains are considerable even for moderate values of the parameter γ.

The simplest variant of problem (1.1)–(1.3) corresponds to the case where ω is a strip and the heat flux on its boundaries is constant and equal to q. In this case, the functions h and T depend only on one variable $x = x_1$ and Eqs. (1.1) admit single integration. As a result, problem (1.1), (1.3) reduces to determining a positive function $h(x)$ which satisfies the equation

$$h^2 h^{\mathrm{III}} = -b, \qquad 0 < x < 1 \tag{1.7}$$

and the additional relations

$$\dot{h}(0) = \dot{h}(1) = 0; \tag{1.8}$$

$$\int_0^1 h(x)\, dx = 1. \tag{1.9}$$

Here $b = -\gamma q$; the dot denotes differentiation with respect to x. Without loss of generality, we can assume that the number b is nonnegative (the case $b \leqslant 0$ reduces to the previous one by the substitution $\tilde{x} = 1 - x$). After the solution of problem (1.7)–(1.9), the function $T(x)$ is determined from the formula

$$T = -q \int_0^1 \frac{dy}{h(y)} + q \int_0^1 \Big(\int_0^x \frac{dy}{h(y)} \Big)\, dx. \tag{1.10}$$

Relation (1.10) was derived taking into account the normalization condition (1.5).

A preliminary study of problem (1.7)–(1.9) was performed in [11]. It was established that, for $b > 0$, the function $h(x)$ increases strictly monotonically in the interval $(0, 1)$ and has a single point of inflection x_*. If $b = 0$, the unique solution of problem (1.7)–(1.9) is $h = 1$. For small b, the solution of this problem has the asymptotics

$$h = 1 + b(-1/24 + x^2/4 - x^3/6) + O(b^2), \qquad b \to 0. \tag{1.11}$$

In addition, it was proved that, for any finite $b > 0$, the quantity $h(0)$ is positive and $\ddot{h}(0) > 0$. Below, it is proved that problem (1.7)–(1.9) has at least one solution for any $b > 0$, and results of its numerical investigation are given.

2. Solvability of Problem (1.7)–(1.9). Effective investigation of problem (1.7)–(1.9) is based on the reduction of Eq. (1.7) to a first-order equation. This can be done because the indicated equation has a two-parameter group of transformations (translation along x and uniform extension of the variables x and h). Using these symmetry properties, the examined problem can be reduced to the Cauchy problem

$$y^2 y^{\mathrm{III}} = -1; \tag{2.1}$$

$$y = 1, \qquad \dot{y} = c, \qquad \ddot{y} = 0 \qquad \text{at} \qquad x = 0, \tag{2.2}$$

where c is a positive constant. We prove that, for any $c > 0$, problem (2.1), (2.2) has a solution with the following properties:

1) there exist $x_1 < 0$ and $x_2 > 0$ such that $\dot{y}(x_1) = \dot{y}(x_2) = 0$;
2) $y(x) > 0$ if $x_1 \leqslant x \leqslant x_2$;
3) $\dot{y}(x) > 0$ if $x_1 < x < x_2$;
4) $\ddot{y}(x) > 0$ if $x_1 \leqslant x < 0$, and $\ddot{y}(x) < 0$ if $0 < x \leqslant x_2$.

In Eq. (2.1) we transform to a new independent variable s and a new required function z using the formulas

$$y = \exp(s), \qquad \dot{y} = z(s). \tag{2.3}$$

The function $z(s)$ satisfies the second-order equation

$$z^2\Big(\frac{d^2z}{ds^2}-\frac{dz}{ds}\Big)+z\Big(\frac{dz}{ds}\Big)^2=-1 \tag{2.4}$$

and the initial conditions

$$z=c, \qquad \frac{dz}{ds}=0 \qquad \text{at} \qquad s=1.$$

Substitution of the expression

$$\Big(z\,\frac{dz}{ds}\Big)^2=w(z) \tag{2.5}$$

into (2.4) reduces Eq. (2.4) to the first-order equation

$$\Big(\frac{dw}{dz}+2\Big)^2=z^2w,$$

which is equivalent to the two equations

$$\frac{dw_1}{dz}=-2-zw_1^{1/2}; \tag{2.6}$$

$$\frac{dw_2}{dz}=-2+zw_2^{1/2}. \tag{2.7}$$

By the definition, the functions w_1 and w_2 cannot take negative values. For Eqs. (2.6) and (2.7), we examine the Cauchy problems

$$w_k(c)=0, \qquad k=1,2. \tag{2.8}$$

The smoothness of the right sides of Eqs. (2.6) and (2.7) is violated at the point $z=c$, $w_k=0$. However, both Cauchy problems (2.6), (2.8) and (2.7), (2.8) have unique solutions defined in the left half-neighborhood of the point $z=c$, where the asymptotics of the functions $w_k(z)$ has the form

$$w_1=2(c-z)+(c/3)[2(c-z)]^{3/2}+O(c-z)^2, \quad w_2=2(c-z)-(c/3)[2(c-z)]^{3/2}+O(c-z)^2.$$

The integral curve of Eq. (2.7) with origin at the point $z=c$, $w_1=0$ is above the straight line $w_1=2(c-z)$. Since the right side of this equation increases sublinearly along the variable w_1, the solution of the Cauchy problem (2.7), (2.8) can be continued up to the point $z=0$, and $w_1(0)>2c$.

The integral curve of Eq. (2.6) with origin at the point $z=c$, $w_2=0$ is below the straight line $w_2=2(c-z)$. Reasoning by contradiction shows that, for $0\leqslant z<0$, the equality $w_2(z)=0$ is impossible. Continuing the solution w_2 to its natural boundary $z=0$, we conclude that $0<w_2(0)<2c$.

If the solutions of problems (2.6)–(2.8) are known, the functions $s_1(z)$ and $s_2(z)$ can be determined by the relations

$$s_1(z)=-\int\limits_z^c\frac{\zeta\,d\zeta}{[w_1(\zeta)]^{1/2}}, \qquad s_2(z)=\int\limits_z^c\frac{\zeta\,d\zeta}{[w_2(\zeta)]^{1/2}}, \qquad 0\leqslant z\leqslant c. \tag{2.9}$$

In the interval $[0,c)$, the function s_1 increases strictly monotonically, and the function s_2 decreases strictly monotonically. Convergence of the integrals in (2.9) is guaranteed by the fact that both functions w_1 and w_2 have the main linear part $2(c-z)$ as $z\to c-0$. Inverting relations (2.9) with respect to z, we obtain two functions: $z=Z_1(s_1)$ and $z=Z_2(s_2)$. The first of these functions is determined on the segment $\xi_1\leqslant s_1\leqslant 0$ and increases monotonically from zero to c, and the second is determined on the segment $0\leqslant s_2\leqslant\xi_2$ and decreases monotonically from c to zero. Here $\xi_k=s_k(0)$, $k=1,2$. The functions $Z_k(s_k)$ satisfy Eq. (2.5) with the right side w_k ($k=1,2$). These functions are analytical in the intervals $(\xi_1,0)$ and $(0,\xi_2)$, and, at the ends of the indicated intervals, they have the asymptotics

$$Z_k=[2w_k(0)|s_k-\xi_k|]^{1/2}+O(|s_k-\xi_k|)^{3/2} \qquad \text{at} \qquad s_k\to\xi_k, \quad k=1,2.$$

In addition, the function Z_2^2 is an analytical continuation of the function Z_1^2 from the interval $(\xi_1,0)$ to the interval $(0,\xi_2)$.

We obtain a parametric representation of the solution of the Cauchy problem in the intervals $[x_1, 0]$ and $[0, x_2]$ in terms of integrals of the functions $w_k(z)$ [we recall that $\dot{y}(x_1) = \dot{y}(x_2) = 0$)]. From (2.3) it follows that $dx = \exp(s)[z(s)]^{-1}\,ds$. In view of (2.9) and the relation $y = \exp(s)$, this makes it possible to determine the dependences of x and y on z using the relations

$$\begin{gathered} y = \exp\Big(-\int\limits_z^c \frac{\zeta\,d\zeta}{[w_1(\zeta)]^{1/2}}\Big), \quad x = -\int\limits_z^c \exp\Big(-\int\limits_\zeta^c \frac{\eta\,d\eta}{[w_1(\eta)]^{1/2}}\Big)\frac{d\zeta}{[w_1(\zeta)]^{1/2}} \quad \text{at} \quad x_1 \leqslant x \leqslant 0, \\ y = \exp\Big(\int\limits_z^c \frac{\zeta\,d\zeta}{[w_2(\zeta)]^{1/2}}\Big), \quad x = \int\limits_z^c \exp\Big(\int\limits_\zeta^c \frac{\eta\,d\eta}{[w_2(\eta)]^{1/2}}\Big)\frac{d\zeta}{[w_2(\zeta)]^{1/2}} \quad \text{at} \quad 0 \leqslant x \leqslant x_2. \end{gathered} \tag{2.10}$$

The values of $x_1(c)$ and $x_2(c)$ are obtained from formulas (2.10) if we set $z = 0$ in them. It is easy to verify that, for an arbitrary value of the parameter $c > 0$, the function $y(x)$ determined implicitly by equalities (2.10) satisfies Eq. (2.1) and the initial conditions (2.2) and also have properties 1–4.

In Eq. (2.1), we transform to a new independent variable $\tilde{x}$ and a new required function $h(\tilde{x})$:

$$x = (x_2 - x_1)\tilde{x} + x_1, \qquad y(x) = Ah(\tilde{x}). \tag{2.11}$$

Then, the interval $x_1 < x < x_2$ becomes the interval $0 < \tilde{x} < 1$. We require that the function $h(\tilde{x})$ satisfy condition (1.9). This implies the following relation between the quantities x_1, x_2, A, and c:

$$(x_2 - x_1)A = \int\limits_{x_1}^{x_2} y(x)\,dx. \tag{2.12}$$

Due to (2.11), the function $h(\tilde{x})$ determined for $0 \leqslant x \leqslant 1$ satisfies the equation

$$h^2\,\frac{d^3h}{d\tilde{x}^3} = -A^{-3}(x_2 - x_1)^3.$$

Based on (2.10) and (2.11), this function also satisfies boundary-conditions (1.8). In order that the last equation coincide with (1.7), it is necessary that the equality $(x_2 - x_1)^3 = bA^3$ be satisfied. Substituting the values of x_1, x_2, and A found by formulas (2.10) and (2.12) into this equality and transforming to the variable z in the calculation of integral (2.12), we obtain the equation relating the parameters b and c:

$$\begin{gathered} F(c) \equiv \Big[\int\limits_0^c \sum_{k=1}^2 \exp\Big((-1)^k \int\limits_z^c \frac{\zeta\,d\zeta}{[w_k(\zeta)]^{1/2}}\Big)\frac{dz}{[w_k(z)]^{1/2}}\Big]^2 \\ \times \Big[\int\limits_0^c \sum_{k=1}^2 \exp\Big(2(-1)^k \int\limits_z^c \frac{\zeta\,d\zeta}{[w_k(\zeta)]^{1/2}}\Big)\frac{dz}{[w_k(z)]^{1/2}}\Big]^{-1} = b^{1/3}. \end{gathered} \tag{2.13}$$

The aforesaid implies that, in Eq. (2.13), the function $F(c)$ is definite and continuous for any $c > 0$. Thus, solving the Cauchy problem (2.1), (2.2) for various values c, we obtain the solution of the initial boundary-value problem (1.7)–(1.9) for values of the parameter $b = F(c)$. If c is small, the solutions of both Cauchy problems (2.6), (2.8) and (2.7), (2.8) have identical asymptotics:

$$w_k = 2(c - z) + O(c - z)^{3/2}, \qquad 0 \leqslant z \leqslant c, \quad k = 1, 2, \quad c \to 0.$$

Substituting these expressions into Eq. (2.13), we obtain $F = 2(2c)^{1/2} + O(c)$ with $c \to 0$. If one proves that $F \to \infty$ as $c \to \infty$, this implies the solvability of Eq. (2.13) for any $b > 0$.

An analysis of the behavior of the function F for large values of c is difficult to because the asymptotics of the functions w_1 and w_2 are not uniform over the entire interval $[0, c]$ if $c \to \infty$. We first examine problem (2.6), (2.8). Transforming to new variables $z = ct$ and $w_1(z) = c^4 u(t)$, we obtain

$$\frac{du}{dt} = -\varepsilon - tu^{1/2}, \qquad 0 < t < 1, \quad u(1) = 0, \tag{2.14}$$

where $\varepsilon = 2c^{-3}$. In any interval $[0, 1 - \varepsilon]$, the solution of problem (2.14) has the representation

$$16u = (1 - t^2)^2 + O(\varepsilon), \qquad \varepsilon \to 0. \tag{2.15}$$

In the interval $[1-\varepsilon, 1]$, the main term u_0 of the asymptotics of the function u is determined implicitly from the equation

$$2u_0^{1/2} - 2\varepsilon \ln(1 + \varepsilon^{-1} u_0^{1/2}) = 1 - t, \qquad \varepsilon \to 0. \tag{2.16}$$

Representations (2.15) and (2.16) are sufficient to obtain the estimates

$$\int_0^c \exp\Big(-n \int_z^c \frac{\zeta\, d\zeta}{[w_1(\zeta)]^{1/2}}\Big) \frac{dz}{[w_1(z)]^{1/2}} = O(1), \qquad c \to \infty, \quad n = 1, 2. \tag{2.17}$$

We now examine problem (2.7), (2.8). The function $v(t) = c^{-4} w_2(z)$ is a solution of the Cauchy problem

$$\frac{dv}{dt} = -\varepsilon + t v^{1/2}, \qquad 0 < t < 1, \quad v(1) = 0.$$

In the interval $[0, 1-\varepsilon]$, the asymptotics of solution of this problem is written as

$$v = \varepsilon^2 + O(\varepsilon^3), \qquad \varepsilon \to 0. \tag{2.18}$$

In the interval $[1-\varepsilon, 1]$, the main term v_0 of the asymptotics of the function v satisfies the equation

$$-2v_0^{1/2} - 2\varepsilon \ln(1 - \varepsilon^{-1} v_0^{1/2}) = 1 - t, \qquad \varepsilon \to 0. \tag{2.19}$$

In this case, the inequality $v_0 < \varepsilon^2$ holds for $t < 1$. Hence, in Eq. (2.19), the argument of the logarithmic function is positive. Formulas (2.18) and (2.19) lead to the representations

$$\int_0^c \exp\Big(\int_z^c \frac{n\zeta\, d\zeta}{[w_2(\zeta)]^{1/2}}\Big) \frac{dz}{[w_2(z)]^{1/2}} = (2^{1-n}\pi c)^{1/2} \exp\Big(\frac{nc^3}{4}\Big)[1 + O(c^{-1})], \tag{2.20}$$
$$c \to \infty, \qquad n = 1, 2.$$

Estimates (2.19) were derived taking into account that the main contribution to the estimated integrals comes from the values of the subintegral function in the interval $[0, c - c\varepsilon]$, where the function w_2 is close to the constant $4c^{-2}$.

From the results of comparison of expressions (2.17) and (2.20), it follows that, for large values of c, the main contribution to the function $F(c)$ comes from the second terms of the sums in both the numerator and denominator of the fraction defining this function. Separating the main term of the asymptotics of $F(c)$, we obtain $F = (2\pi c)^{1/2} + O(1)$ if $c \to \infty$. Thus, the function $F(c)$ defined for any $c \geqslant 0$ has the following properties: 1) it is continuous; 2) it takes positive values for $c > 0$; 3) $F(0) = 0$; 4) $F \to \infty$ if $c \to \infty$. From this it follows that the equation $F(c) = b$ has at least one solution for any $b > 0$. Thus, it is proved that problem (1.7)–(1.9) is solvable for any positive value of the parameter b proportional to the dimensionless heat flux q. Because $q > 0$, it follows that, the temperature decreases with increasing x while the film thickness increases as the cooled boundary $x = 1$ is approached. This effect is natural since the surface tension coefficient increases with decreasing temperature.

3. Numerical Solution of Problem (1.7)–(1.9). We will solve the problem using two methods: summation of power series and the collocation method.

We seek a solution in the form of the power series

$$h(x) = 1 + \sum_{j=1}^{\infty} h_j(x) b^j. \tag{3.1}$$

Here h_1 it is known [see (1.11)]. We find the coefficients h_j for $j > 1$. Substituting series (3.1) into problem (1.7)–(1.9) and equating the terms at identical powers of b, we obtain the following sequence of boundary-value problems for the third-order linear differential equation:

$$a_{j-1} = \sum_{k=0}^{j-1} h_k h_{j-k}, \qquad h_j^{\rm III} = -\sum_{k=1}^{j-1} h_k^{\rm III} a_{j-k},$$
$$\dot h_j(0) = 0, \qquad \dot h_j(1) = 0, \qquad \int_0^1 h_j\, dx = 0. \tag{3.2}$$

Solving problem (3.2) recursively for each $j > 1$, we find a pair of functions $a_{j-1}(x)$ and $h_j(x)$. As a result, we have

$$h_2 = -\frac{1}{3360} + \frac{1}{120}x^2 - \frac{1}{72}x^3 + \frac{1}{120}x^5 - \frac{1}{360}x^6,$$

$$h_3 = -\frac{473}{7{,}257{,}600} + \frac{17}{26{,}880}x^2 - \frac{13}{13{,}440}x^3 + \frac{19}{14{,}400}x^5 - \frac{1}{1728}x^6 - \frac{1}{1120}x^7 + \frac{1}{1260}x^8 - \frac{1}{5670}x^9.$$

All subsequent coefficients are similar polynomials of x, whose power increases rapidly with increasing coefficient number.

Let us estimate the radius of convergence of series (3.1). We note the well-known fact: in an arbitrary power series, if there exists a limit for the ratio of the previous coefficient to the subsequent coefficient as the coefficient numbers tend to infinity, this limit is equal to the singular point of the function represented by the power series considered. For (3.1), such a limit does not exist. However, if a series $\sum h_j b^j$ (here h_j are numbers) is broken into even and odd components $\sum h_{2j} b^{2j}$ and $\sum h_{2j+1} b^{2j+1}$, then for these series there exist identical limits of sequences

$$\lim_{j\to\infty} \sqrt{h_{2j-2}/h_{2j}} = \lim_{j\to\infty} \sqrt{h_{2j-1}/h_{2j+1}} = b^*.$$

This implies that, for $b = b^*$, the function $h(x)$ has a singularity which limits the convergence, i.e., the radius of convergence (3.1) is equal to b^*.

Below, for various values of x, we give the last five elements of the sequences

$$\left\{\sqrt{h_{2j-2}/h_{2j}}\right\}, \qquad \left\{\sqrt{h_{2j-1}/h_{2j+1}}\right\} \tag{3.3}$$

(the maximum index of the coefficient used is 50). For example, for $x = 0$, we have the following power series:

$$h = 1 - \frac{1}{24}b - \frac{1}{3360}b^2 - \frac{473}{7{,}257{,}600}b^3 - \frac{33{,}023}{29{,}059{,}430{,}400}b^4 + \dots\ .$$

The corresponding sequences (3.3) have the form

9.665,309,932	9.675,477,093
9.648,750,827	9.657,961,890
9.633,693,982	9.642,077,742
9.619,943,821	9.627,606,945
9.607,337,313	9.614,368,880
....	....

This suggests that both sequences tend to the same limit. Similarly, for $x = 1/2$, we have a different power series:

$$h = 1 + \frac{43}{161{,}280}b^2 + \frac{2{,}040{,}169}{2{,}231{,}764{,}254{,}720}b^4 + \frac{47{,}418{,}886{,}356{,}551}{8{,}719{,}520{,}797{,}305{,}077{,}760{,}000}b^6 + \dots\ .$$

Here the odd component is absent; therefore, we can form just one sequence:

9.667,366,237
9.650,615,617
9.635,392,814
9.621,497,904
9.608,764,387
....

This sequence tends to approximately the same limit. Results of similar studies at other points x suggest that the radius of convergence of series (3.1) does not depend on x. For $b = b^* \approx 9.3$, the solution has a singular point. Obviously, this is a bifurcation point. Constructing the solution for $b > b^*$ requires special methods.

Figure 1 shows film thickness profiles for nine integer values of b obtained using 15 terms of the power series (3.1) (solid curve). As the parameter b increases, the heat flux on the boundary increases and, accordingly, the film strain increases.

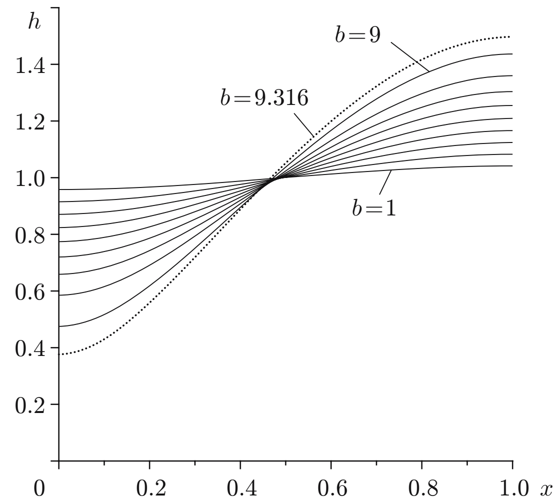

Fig. 1. Numerical solution of the one-dimensional problem (1.7)–(1.9) for various values of the parameter b: the solid curve refers to the calculation using the series summation method and the collocation method; the dotted curve refers to the calculation using the collocation method ($b = 9.316$ is the bifurcation point).

To control the error, we use the collocation method. We seek a solution in the form of a polynomial of the Nth power:

$$h(x) = c_1 + c_2 x^2 + \ldots + c_N x^N. \tag{3.4}$$

The approximation accuracy increases with increasing number N. Polynomial (3.4) does not contain a term with the first power; therefore, the left boundary-value condition (1.8) is satisfied automatically. The quantities to be determined are the coefficients of the polynomial $c_1, c_2, \ldots, c_N$. A solution of the form (3.4), generally speaking, does not satisfy Eq. (1.7). The essence of the collocation method consists of requiring that this equation be valid at a finite number of points. Substituting solution (3.4) into (1.7) and requiring its validity at $N - 2$ points $x_i = (i-1)/(N-3)$, we obtain the equations

$$f_i(c_1, c_2, \ldots, c_N) = 0 \qquad (i = 1, 2, \ldots, N-2). \tag{3.5}$$

From the right boundary-value condition (1.8) and the integral relation (1.9), we obtain two more equations, respectively:

$$2c_1 + 3c_2 + \ldots + Nc_N = 0; \tag{3.6}$$

$$c_1 + \frac{c_2}{3} + \frac{c_3}{4} + \ldots + \frac{c_N}{N+1} - 1 = 0. \tag{3.7}$$

As a result, we have a system of N nonlinear equations for N unknowns $c_1, c_2, \ldots, c_N$. System (3.5)–(3.7) was solved using the Newton method. The solution $h = 1$ was taken to be the initial approximation for small b. Next, analytical continuation in the parameter b (by small steps in b toward its increase) was performed, and, in each step, the solution obtained in the previous step was used as the initial approximation. If the step is small enough and the Jacobian of the system of algebraic equations is nonzero, the Newton method always converges with this choice of the initial approximation. The results obtained using the collocation method coincide with the results found by summation of the power series (3.1).

For integer b, the film thickness profiles coincided (with an error smaller than the line thickness) with the profiles in Fig. 1 (solid curve). The dotted curve in Fig. 1 corresponds to the profile with the largest calculated strain obtained for $b \approx b^*$ only using the collocation method since the power series (3.1) diverges. In the Newton method for $b \approx b^*$, the Jacobian vanishes (in the real calculations, the change in the sign of the Jacobian was fixed). This means that the bifurcation point is found. For $b > b^*$, the further analytical continuation proved impossible because the iterations in the Newton method ceased to converge. In this case, one needs an additional investigation of the character of the bifurcation point and a special choice of the initial approximation in the Newton method. With a further increase in b, the film strain continues to increase, and to construct the film thickness profile, it is necessary to use special methods.

4. Numerical Solution of Problem (1.1)–(1.3) in a Circle. We seek a solution of problem (1.1)–(1.3) in the form of series (1.6). Substituting these series into the equations and equating terms at identical powers of γ, we obtain a sequence of boundary-value problems. To find a pair of functions T_0 and h_1, it is required to solve the following problem:

$$\Delta T_0 = 0, \qquad \Delta\Delta h_1 = 0, \qquad (x_1, x_2) \in \omega,$$
$$\frac{\partial T_0}{\partial n} = g, \qquad \frac{\partial h_1}{\partial n} = 0, \qquad \frac{\partial \Delta h_1}{\partial n} = g, \qquad (x_1, x_2) \in \partial\omega,$$
$$\int\limits_\omega T_0\, d\omega = 0, \qquad \int\limits_\omega h_1\, d\omega = 0.$$

The subsequent functions T_k, h_{k+1} $(k = 1, 2, \ldots)$ are determined from the recursive system of equations and boundary conditions:

$$\Delta T_k = -\nabla \cdot \Big(\sum_{i=1}^{k} h_i \nabla T_{k-i}\Big),$$
$$\Delta\Delta h_{k+1} = \Delta T_k - \nabla\Big(\sum_{i=1}^{k} h_i \nabla \Delta h_{k+1-i}\Big), \qquad (x_1, x_2) \in \omega,$$
$$\frac{\partial T_k}{\partial n} = -\sum_{i=1}^{k} h_i \frac{\partial T_{k-i}}{\partial n}, \qquad \frac{\partial h_{k+1}}{\partial n} = 0,$$
$$\frac{\partial \Delta h_{k+1}}{\partial n} = \frac{\partial T_k}{\partial n} - \sum_{i=1}^{k} h_i \frac{\partial \Delta h_{k+1-i}}{\partial n}, \qquad (x_1, x_2) \in \partial\omega,$$
$$\int\limits_\omega T_k\, d\omega = 0, \qquad \int\limits_\omega h_{k+1}\, d\omega = 0.$$

Solving these boundary-value problems sequentially, in polar coordinates ($x_1 = r\cos\varphi$ and $x_2 = r\sin\varphi$), we obtain

$$T_0(r,\varphi) = \frac{1}{2}\, r^2 \sin 2\varphi, \qquad h_1(r,\varphi) = \frac{1}{24}\,(r^4 - 2r^2)\sin 2\varphi,$$
$$T_1(r,\varphi) = -\frac{1}{288}\, r^6 + \frac{1}{96}\, r^4 - \frac{1}{384} + \Big(\frac{1}{480}\, r^6 - \frac{1}{120}\, r^4\Big)\cos 4\varphi,$$
$$h_2(r,\varphi) = -\frac{1}{9216}\, r^8 + \frac{1}{1728}\, r^6 - \frac{1}{768}\, r^2 + \frac{73}{138{,}240} + \Big(\frac{1}{11{,}520}\, r^8 + \frac{31}{28{,}800}\, r^4 - \frac{1}{1200}\, r^6\Big)\cos 4\varphi,$$
$$T_2(r,\varphi) = \Big(\frac{71}{552{,}960}\, r^{10} - \frac{31}{43{,}200}\, r^8 + \frac{221}{230{,}400}\, r^6 + \frac{1}{4608}\, r^4 - \frac{547}{2{,}764{,}800}\, r^2\Big)\sin 2\varphi$$
$$+ \Big(-\frac{1}{36{,}864}\, r^{10} + \frac{31}{201{,}600}\, r^8 - \frac{67}{258{,}048}\, r^6\Big)\sin 6\varphi,$$
$$h_3(r,\varphi) = \Big(\frac{41}{15{,}482{,}880}\, r^{12} - \frac{7}{331{,}776}\, r^{10} + \frac{79}{1{,}728{,}000}\, r^8 + \frac{1}{73{,}728}\, r^6$$
$$-\frac{1181}{33{,}177{,}600}\, r^4 - \frac{4049}{64{,}512{,}000}\, r^2\Big)\sin 2\varphi + \Big(-\frac{29}{39{,}813{,}120}\, r^{12} + \frac{29}{4{,}300{,}800}\, r^{10} - \frac{1109}{43{,}352{,}064}\, r^8 + \frac{23{,}729}{975{,}421{,}440}\, r^6\Big)\sin 6\varphi.$$

The subsequent terms of the series are similar trigonometric polynomials, whose coefficients, in turn, are polynomials of r.

Let us estimate the radius of convergence of these series. For example, for $r = 0$, we have

$$h = 1 + \frac{73}{138\,240}\,\gamma^2 + \frac{238{,}291{,}217}{156{,}067{,}430{,}400{,}000}\,\gamma^4 + \frac{86{,}683{,}166{,}091{,}871{,}277{,}713}{10{,}057{,}327{,}448{,}227{,}332{,}489{,}216{,}000{,}000}\,\gamma^6 + \dots\,.$$

All coefficients of this power series are positive numbers. According to the well-known theorem of complex analysis, this implies that the singularity bounding the convergence of the series and located in the complex plane γ is on the positive real axis. To find this singular point, we form, as above, a sequence of square roots of the ratio of the previous of the subsequent coefficient (the maximum number of the coefficient used is 20; the last five elements of the sequence are given):

10.833,340,12
10.576,530,24
10.395,380,35
10.260,786,14
10.156,865,19
. . . .

It is known that, if this sequence converges, it converges to the radius of convergence. Obviously, the radius of convergence is a number that slightly exceeds 10. What is the radius of convergence at other points of the circle? For example, for $r = 1$ and $\varphi = 0$, we have

$$h = 1 + \frac{1}{38{,}400}\,\gamma^2 + \frac{892{,}430{,}023}{5{,}407{,}736{,}463{,}360{,}000}\,\gamma^4 + \frac{781{,}713{,}706{,}853{,}446{,}010{,}687{,}833}{72{,}838{,}727{,}350{,}200{,}255{,}733{,}825{,}536{,}000{,}000}\,\gamma^6 + \dots\,.$$

Similarly, the sequence

10.705,057,91
10.488,521,95
10.331,243,72
10.211,961,27
10.118,448,70
. . . .

converges to the same limit. Considering the point $r = 1$, $\varphi = \pi/4$, we obtain the power series containing both even and odd powers:

$$h = 1 - \frac{1}{24}\,\gamma - \frac{73}{115{,}200}\,\gamma^2 - \frac{379{,}741}{6{,}096{,}384{,}000}\,\gamma^3 - \frac{10{,}223{,}842{,}219}{4{,}326{,}189{,}170{,}688{,}000}\,\gamma^4 + \dots\,.$$

In this case, it is possible to form two sequences similar to (3.3):

10.734,581,21	10.924,129,72
10.505,732,21	10.635,848,20
10.342,024,02	10.436,932,87
10.219,072,88	10.291,393,49
10.123,325,62	10.180,282,35
. . . .	. . . .

Each of these sequences tends to the same limit.

A detailed study for various points of the unit circle shows that, for the functions T and h, the radii of convergence, first, are identical, and second, do not depend on r and φ. In other words, all parts of the solution represented by power series fail simultaneously for $\gamma = \gamma^* \approx 10.1$. It was assumed that this failure occurs when the film thickness becomes equal to zero, but this failure occurs earlier.

How much rapidly do the series converge and how many terms of the series is it necessary to take to obtain reasonable accuracy of the solution up to $\gamma = 10$? We examine this question for $r = 1$. In this case, the solution for the film thickness is written as

$$h = 1 - \frac{1}{24}\,\sin 2\varphi\,\gamma + \Big(-\frac{7}{23{,}040} + \frac{19}{57{,}600}\,\cos 4\varphi\Big)\gamma^2 + \Big(\frac{5567}{96{,}768{,}000}\,\sin 2\varphi + \frac{1451}{304{,}819{,}200}\,\sin 6\varphi\Big)\gamma^3 + \dots\,.$$

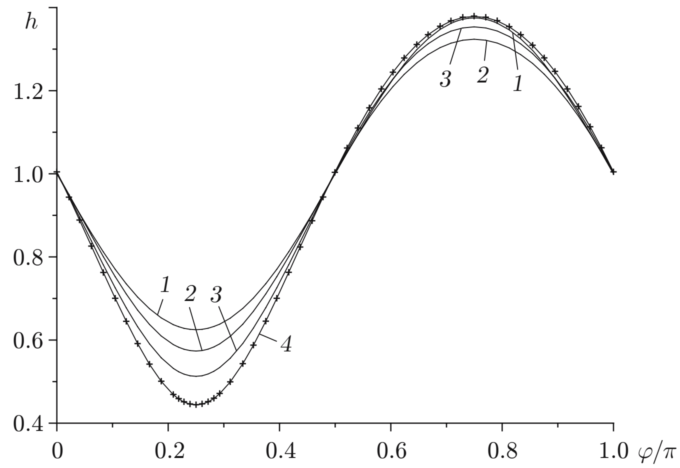

Fig. 2. Partial sums of series (1.6) for the film thickness profile on a circle $r = 1$ for $\gamma = 9$ and various number of the terms kept N: $N = 1$ (1), 2 (2), 4 (3), and 20 (4); the points refer to $N = 19$.

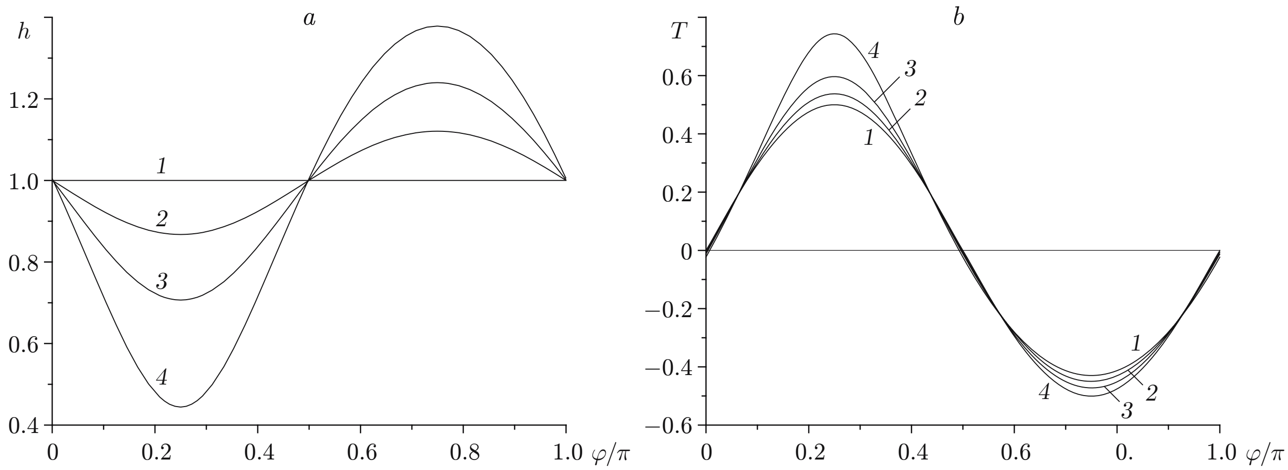

Fig. 3. Variation of the film thickness (a) and temperature (b) for $r = 1$: $\gamma = 0$ (1), 3 (2), 6 (3), and 9 (4).

In this series, we keep terms up to the terms γ^N inclusive. The obtained partial sums are presented in Fig. 2 for $\gamma = 9$ and various values of N. Figure 2 shows the convergence of series (1.6). The points corresponding to the value $N = 19$ almost coincide with the film thickness profile corresponding to $N = 20$. From Fig. 2 and from the results of a similar study for the temperature profile, it follows that 20 terms of the series are sufficient to describe the solution everywhere with high accuracy almost up to the radius of convergence.

The evolution of the film thickness and temperature profile for $r = 1$ and various values of γ is presented in Fig. 3. It is evident that an increase in the heat flux on the boundary (i.e., with an increase in the parameter γ) leads to a considerable strain of the film and an increase in the temperature gradient. Construction of solutions for larger values of the parameter γ by direct summation of series (1.6) is impossible since, for $\gamma > 10$, the series begin to diverge.

The shape of the film surface for $\gamma = 10$ is given in Fig. 4. For illustration, the scales on the x_1, x_2, and h axes are different. Constant level lines are shown on the film surface. For this value of the heat flux on the cylinder, the flow is subjected to considerable strains and, at two points, the film thickness, becomes small. Obviously, a further increase in γ leads to rupture of the film at these points.

Figure 5 shows lines of constant temperature T for $\gamma = 10$ in the plane (x_1, x_2). The darker sites correspond to higher temperature. The temperature field on the film is generally smoother than that on its boundary. In other words, the maximum temperature gradients are reached on the circle.

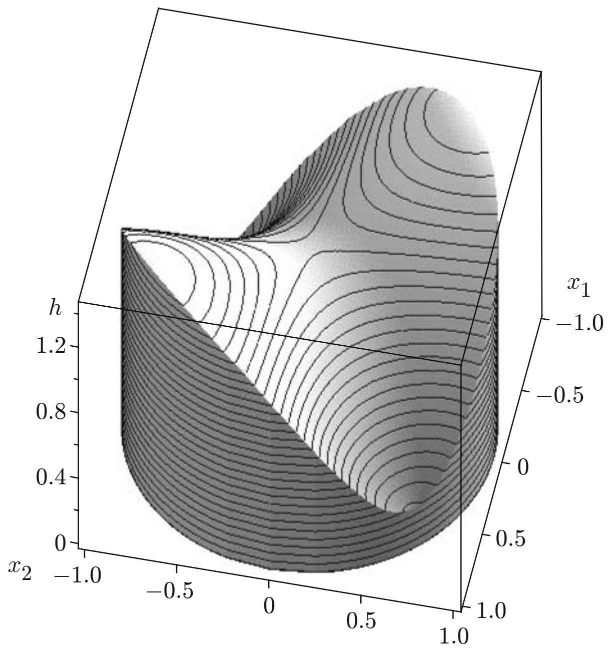

Fig. 4

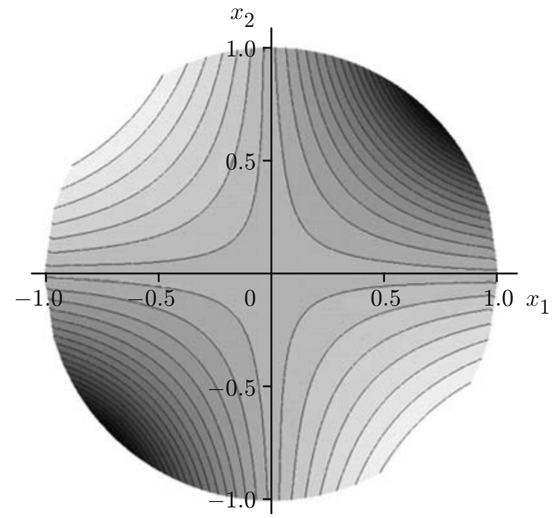

Fig. 5

Fig. 4. Thickness profile h of a film located in a nonuniformly heated cylinder for $\gamma = 10$.

Fig. 5. Temperature level lines for a film fixed inside a solid cylinder of unit radius.

This work was supported by the Program of the President of the Russian Federation on the State Support of Leading Scientific Schools (Grant NSh-2260.2008.1) and the Integration project of the Siberian Division of the Russian Academy of Sciences No. 111.

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